

Final Report for Period: 06/2002 - 05/2006

Submitted on: 08/30/2006

Principal Investigator: Monteiro, Renato D. C.

Award ID: 0203113

Organization: GA Tech Res Corp - GIT

Title:

Collaborative Research: Theory and Implementation of Semidefinite Programming and its Applications to Combinatorial Optimization

Project Participants

Senior Personnel

Name: Monteiro, Renato D.C.

Worked for more than 160 Hours: Yes

Contribution to Project:

Post-doc

Graduate Student

Name: Lu, Zhaosong

Worked for more than 160 Hours: Yes

Contribution to Project:

Zhaosong Lu has helped me to generate important ideas which have been invaluable for the project. He has also played a fundamental role in the implementation part of the project. This project also serves as the basis for his Ph.D. dissertation.

Zhaosong graduated in August of 2005 and is currently Assistant Professor in the Department of Mathematics at the Simon Fraser University in Canada.

Name: O'Neal, Jerome

Worked for more than 160 Hours: Yes

Contribution to Project:

Jerome O' Neal received only travel support from this grant. His monthly income was taken care by other sources. He was allowed to work on an project that would provide him with a good topic for his Ph.D. thesis. Jerome's main research work dealt with second-order interior-point methods for linear and convex quadratic programming based on iterative solvers. He graduated in August of 2005 and went to work for Delta Technology, Inc.

Name: Ekici, Ali

Worked for more than 160 Hours: Yes

Contribution to Project:

Undergraduate Student

Technician, Programmer

Other Participant

Research Experience for Undergraduates

Organizational Partners

Other Collaborators or Contacts

I have been collaborating with the following researchers during the course of this grant:

Takashi Tsuchiya (Institute of Statistical Math in Japan)
 Yin Zhang (Rice University - USA)
 Sam Burer (University of Iowa)
 Joao X. da Cruz Neto (Universidade Federal do Piaui, Brazil)
 Orizon P. Ferreira (Universidade Federal de Goiás, Brazil)
 Alfredo Iusem (IMPA, Rio de Janeiro, Brazil)
 Arkadi Nemirovski (Georgia Tech, USA)
 Ming Yuan (Georgia Tech, USA)

Activities and Findings

Research and Education Activities: (See PDF version submitted by PI at the end of the report)

Findings: (See PDF version submitted by PI at the end of the report)

Training and Development:

This grant has partially supported two Ph.D. students, namely Jerome O'Neal and Zhaosong Lu. Zhaosong Lu started working for this project in the beginning of Spring 2001 and should be graduating in July of 2005. Jerome O'Neal started working on this project in the beginning of Summer 2002 and should be graduating in July of 2005.

In addition to having provided topics for their Ph.D. thesis, this project also gave them the opportunity to travel to conferences and local meetings so that they could be exposed to the research and the researchers in their field of expertise, and consequently promote their career. This grant has partially enabled both students to attend various domestic and international conferences.

Outreach Activities:

Journal Publications

Renato D. C. Monteiro, Jerome W. O'Neal and Takashi Tsuchiya, "Uniform Boundedness of a Preconditioned Normal Matrix Used in Interior Point Methods", SIAM Journal of Optimization, p. 96, vol. 15, (2004). Published

Renato D.C. Monteiro and Takashi Tsuchiya, "A new iteration-complexity bound for the MTY predictor-corrector algorithm", SIAM Journal on Optimization, p. 319, vol. 15, (2004). Published

Renato D. C. Monteiro, "First- and Second-Order Methods for Semidefinite Programming", Mathematical Programming Series B, p. 209, vol. 97, (2003). Published

Renato D. C. Monteiro and Takashi Tsuchiya, "A variant of the Vavasis-Ye layered-step interior-point algorithm for linear programming", SIAM Journal on Optimization, p. 1054, vol. 13, (2003). Published

Sam Burer, Renato D.C. Monteiro and Yin Zhang, "Maximum stable set formulations and heuristics based on continuous optimization", Mathematical Programming, p. 137, vol. 94, (2002). Published

Sam Burer and Renato D.C. Monteiro, "A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization", Mathematical Programming, Series B, p. 329, vol. 95, (2003). Published

Sam Burer, Renato D.C. Monteiro and Yin Zhang, "A computational study of a gradient-based log-barrier algorithm for a class of large-scale SDPs", *Mathematical Programming, Series B*, p. 359, vol. 95, (2003). Published

Zhaosong Lu and Renato D.C. Monteiro, "Error bounds and limiting behavior of weighted paths associated with the SDP map $X^{1/2} S X^{1/2}$ ", *SIAM Journal on Optimization*, p. 348, vol. 15, (2004). Published

Joao X. da Cruz Neto, Orizon P. Ferreira and Renato D.C. Monteiro, "Asymptotic behavior of the central path for a special class of degenerate SDP problems", *Mathematical Programming*, p. 487, vol. 103, (2005). Published

Sam Burer and Renato D.C. Monteiro, "A general framework for establishing polynomial convergence of long-step methods for semidefinite programming", *Optimization Methods and Software*, p. 1, vol. 18, (2003). Published

Sam Burer and Renato D.C. Monteiro, "Local minima and convergence in low-rank semidefinite programming", *Mathematical Programming*, p. 427, vol. 103, (2005). Published

Renato D.C. Monteiro and Jerome W. O'Neal, "Convergence analysis of a long-step primal-dual infeasible interior-point LP algorithm based on iterative linear solvers", *Optimization Methods and Software*, p. , vol. , (). Submitted

Joao X. da Cruz Neto, Orizon P. Ferreira, Alfredo N. Iusem and Renato D.C. Monteiro, "Dual Convergence of the Proximal Point Method with Bregman Distances for Linear Programming", *Optimization Methods and Software*, p. , vol. , (). published electronically

Zhaosong Lu and Renato D.C. Monteiro, "A note on the local convergence of a predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem based on the Alizadeh-Haeberly-Overton search direction", *SIAM Journal on Optimization*, p. 1147, vol. 15, (2005). Published

Zhaosong Lu and Renato D.C. Monteiro, "Limiting behavior of the Alizadeh-Haeberly-Overton weighted paths in semidefinite programming", *Optimization Methods and Software*, p. , vol. , (). Accepted

Z. Lu, R.D.C. Monteiro and J.W. O'Neal, "An iterative solver-based infeasible primal-dual path-following algorithm for convex QP", *SIAM Journal on Optimization*, p. 287, vol. 17, (2006). Published

R.D.C. Monteiro, J.W. O'Neal and A. Nemirovski, "A new conjugate gradient algorithm incorporating adaptive ellipsoid preconditioning", *Optimization Methods and Software*, p. , vol. , (). Submitted

Z. Lu, A. Nemirovski and R.D.C. Monteiro, "Large-scale semidefinite programming via saddle point mirror-prox algorithm", *Mathematical Programming*, p. , vol. , (). Accepted

Z. Lu and R.D.C. Monteiro, "A modified nearly exact method for solving low-rank trust region subproblem", *Mathematical Programming*, p. , vol. , (). Accepted

Z. Lu, R. D. C. Monteiro and J.W. O'Neal, "An iterative solver-based long-step infeasible primal-dual path-following algorithm for convex QP based on a class of preconditioners", *Optimization Methods and Software*, p. , vol. , (). Submitted

R. D. C. Monteiro and T. Tsuchiya, "A strong bound on the integral of the central path curvature and its relation to the iteration-complexity of path following LP algorithms", *Mathematical Programming*, p. , vol. , (). Accepted

M. Yuan, A. Ekici, Z. Lu and R. D. C. Monteiro, "Dimension reduction and coefficient estimation in the multivariate linear regression", *Journal of the Royal Statistical Society, Series B*, p. , vol. , (). Submitted

Books or Other One-time Publications

Web/Internet Site**URL(s):**

www2.isye.gatech.edu/people/faculty/Renato_Monteiro

Description:

This site contains a list of my publications. All the ones cited in this project can be downloaded from the above site.

Other Specific ProductsContributions**Contributions within Discipline:**

Optimization problems, that is the problem of minimizing or maximizing a function of several decision variables subject to the constraint that these variables belong to a certain feasible set, arise in many forms and ways in several branches of science and engineering. Some of these problems can be quite hard to solve but others exhibit sufficient special structure that enable them to be efficiently solved. Semidefinite programming (SDP) is an example of an important class of optimization problems for which powerful solution methodology has been developed to solve it. In a semidefinite programming problem, a linear function of a symmetric matrix variable X is minimized or maximized subject to linear equality constraints on X and the essential constraint that X be positive semidefinite.

The class of SDP problems is quite rich in that several apparently unrelated problems can be reformulated as SDP problems including linear programs, convex quadratic problems with convex quadratic inequality constraints, matrix norm minimization, the problem of finding the ellipsoid of minimum volume that contains a given set of points and ellipsoids, the problem of maximizing the volume of an ellipsoid that is contained in a polytope and a variety of maximum and minimum eigenvalue problems. In addition, SDP has applications in Statistics areas such as minimum trace factor analysis and optimal experiment design, in engineering fields such as structural design, control theory and signal processing, in pattern recognition, in moment problems of probability theory and stochastic optimization, and in finance.

Also, a number of NP-hard combinatorial optimization problems have convex relaxations that are semidefinite programs.

Since the SDP problem can be thought of as a linear program over the cone of positive semidefinite matrices, it is not surprising that it shares many of the nice properties of the standard linear programming problem.

Today, it is known that several interior-point algorithms for linear programs can be extended to SDP problems. As in linear programming, many of these methods are polynomially convergent and perform efficiently in practice. In particular, the class of primal-dual interior-point methods and their higher-order variants are the most effective interior-point methods for solving small- to moderate- sized SDP problems.

Part of this research project was dedicated to the study of the theory of semidefinite programming and the analysis of interior-point primal-dual algorithms for SDP. The investigation of these topics were the subject of previous NSF projects (CCR-9700448 and CCR-9902010) and has continued during the current project. In a series of papers, further results about the theory of semidefinite programming and the polynomial convergence of primal-dual algorithms for SDP have been obtained in this project. More specifically, this project has developed:

- i) new results about the asymptotic behavior of continuous trajectories for SDPs;
- ii) new convergence results on interior-point methods based on iterative solvers;
- iii) new iteration-complexity results for second-order interior-point methods;
- iv) new results about the geometry and curvature of the central path for linear programming.

Another part of this proposal is concerned with nonlinear-programming-type first-order algorithms for solving large-scale SDP problems.

Even though primal-dual interior-point methods are quite robust and require few iterations to converge to obtain highly accurate solutions, an iteration of these methods is expensive and is computationally tractable only for small-to-medium size SDP problems.

In contrast, an iteration of a first-order method is relatively cheap but these methods have the drawback that they require many more iterations to be able to find only low-to-medium accurate solutions.

Presently, among these two classes of algorithms, the class of first-order methods is the only one which can solve large-scale SDPs.

Hence, research development towards understanding and improving the performance of these methods is of paramount importance.

During the duration of this project, as well as the previous NSF grant CCR-9902010,

the P.I. developed in a series of papers powerful first-order methods for solving large-scale SDP problems.

As a result, a code for solving general large-scale SDP code has been implemented and made available to the public (see <http://dollar.biz.uiowa.edu/~bure/software/SDPLR>). This code has been used by several practitioners to solve large-scale SDPs which have originated from relevant problems and applications in their field of interest.

Contributions to Other Disciplines:

Since large-scale semidefinite programming (SDP) problems arise in many applications in continuous optimization, matrix analysis, engineering, statistics and discrete/combinatorial optimization, it is of paramount importance to develop fast and reliable algorithms to solve them.

A few illustrations of the use of SDP in science and engineering are given below. They have been chosen arbitrarily from a list of many other potential applications.

Structural design is an engineering discipline aimed at creating constructions (bridges, cantilevers, the inner skeleton of an airplane wing and etc.) capable of carrying external loads under different loading scenarios. The design of such constructions leads to large-scale optimization problems, called truss topology design (TTD) problem. With the use of duality, this problem can be cast in the form of SDP problems with far fewer constraints than the original problem. This solution methodology has enabled large structural design problems to be routinely solved.

Another important source of applications for SDP lies in the field of system and control theory. Several problems there can be formulated as linear matrix inequalities (LMI) and/or SDP problems. Examples of such problems include matrix scaling problems, construction of quadratic Lyapunov functions for stability and performance analysis of linear differential equations, optimal system realization, inverse problem of optimal control and many others.

In the area of combinatorial optimization, relaxations of NP-hard problems result in SDP problems which are large enough to challenge the efficiency of interior-point methods for solving them.

In this project, the P.I. has provided new insight into the behavior of existing primal-dual interior-point algorithms and developed new interior-point algorithms for solving both the linear and nonlinear SDP problem. The P.I. has also developed novel computational tools that will enable practitioners to solve large-scale SDP problems arising in the types of applications mentioned above. Towards this end, the P.I. has proposed nonlinear-programming-type algorithms for the solution of large-scale specially structured SDP problems, which have performed extremely well on various computational experiments.

In conclusion, this research project has strongly influenced the growing literature on SDP problems and algorithms for solving them. Moreover, it has led to new or improved algorithms to find exact or approximate solutions to large-scale optimization problems arising in diverse applications in industry, finance, science, and engineering.

Contributions to Human Resource Development:

This grant has partially supported four Ph.D. students, namely Jerome O'Neal, Ali Ekici, Guanghui Lan and Zhaosong Lu. Zhaosong Lu graduated in July of 2005 to assume the position of Assistant professor in the Department of Mathematics at the Simon Fraser University in Canada. Jerome O'Neal also graduated on August of 2005 and is currently an Operations Research Specialist in the Dept. of Research, Modeling & Design of Delta Technology, Inc. - Delta Air Lines. Ali Ekici worked with the P.I. for three semesters and decided to pursue another research area under somebody's else supervision. Finally, Guanghui Lan is under the P.I.'s supervision since Summer of 2005.

In addition to having provided topics for their Ph.D. thesis, this project also gave them the opportunity to travel to conferences and local meetings so that they could be exposed to the research and the researchers in their field of expertise, and consequently promote their career.

Contributions to Resources for Research and Education:

Contributions Beyond Science and Engineering:

Categories for which nothing is reported:

Organizational Partners

Activities and Findings: Any Outreach Activities

Any Book

Any Product

Contributions: To Any Resources for Research and Education

Contributions: To Any Beyond Science and Engineering

This project addresses the development of the theory and implementation of primal-dual interior-point algorithms for semidefinite and cone programming problems. More specifically, its objectives consist in:

- 1) advancing the knowledge of the theory of polynomial and superlinear convergence analysis of primal-dual methods for SDP and cone programming;
- 2) studying the existence and asymptotic behavior of continuous trajectories for SDP;
- 3) developing superlinearly convergent higher-order primal-dual interior-point methods for SDP problems without strictly complementary solutions;
- 4) developing interior-point primal-dual algorithms to solve more general classes of cone programming problems;
- 5) developing new and/or improving existing algorithms and implementations for first-order NLP methods for SDP
- 6) implementing these new methods and comparing them against existing SDP methods;
- 7) developing and implementing SDP-based heuristics for solving various NP-hard combinatorial optimization problems
- 8) investigating and exploiting the special structure of SDP relaxations of combinatorial optimization problems
- 9) developing specialized SDP-based implicit enumeration methods for certain combinatorial optimization problems

A total of twenty two papers have been written which acknowledge the grant CCR-0203113. Among these works, eighteen have already appeared or been accepted for publication in the following four major scientific journals: *Mathematical Programming*, *SIAM Journal on Optimization*, *Mathematics of Operations Research* and *Optimization Methods & Software*.

A code for solving general large-scale SDP problems has been implemented and made available to the public (see <http://dollar.biz.uiowa.edu/~bure/software/SDPLR>). This code is a substantially improved version of a previous code, developed by Sam

Burer and the P.I. on an earlier project supported by NSF grant CCR-9902010, which could only solve special types of large-scale SDP graph-relaxation problems.

During the duration of this project, a total of thirty five (35) presentations on the P.I.'s research work were given in major conferences and universities. More specifically, these presentations took place in seven domestic conferences, eight international meeting and eight major US and Canadian universities. Among these presentations, the most influential one was a semi-plenary talk that the P.I. gave at the 18th International Symposium on Mathematical Programming in Copenhagen on August of 2003. The P.I. also gave a plenary lecture in the Brazilian Workshop on Continuous Optimization at Goiania, GO, Brazil in July 19, 2005.

This grant has partially supported four Ph.D. students, namely: Zhaosong Lu, Jerome O'Neal, Ali Ekici and Guanghui Lan. Zhaosong Lu graduated on August of 2005 and is currently Assistant professor in the Department of Mathematics at the Simon Fraser University, Canada. Jerome O'Neal also graduated on August of 2005 and is currently an Operations Research Specialist in the Dept. of Research, Modeling & Design of Delta Technology, Inc. - Delta Air Lines. Even though this grant did not contribute to Jerome O'Neal monthly income, it has supported his travel to attend conferences and give presentations on papers related to this project. Ali Ekici worked with the P.I. for three semesters and decided to pursue another research area under somebody's else supervision. Finally, Guanghui Lan is under the P.I.'s supervision since Summer of 2005.

During the duration of this project, the P.I. have been serving or served as Associate Editor for three journals, namely: *Mathematical Methods of Operations Research* (2002-2005), *INFORMS Journal on Computing* (2001-present) and *Mathematics of Operations Research* (2003-present).

The P.I. have collaborated with eight investigators during the course of this grant, namely: Sam Burer (University of Iowa), Alfredo Iusem (IMPA, Brazil), Joao X. da Cruz Neto (Universidade Federal do Piaui, Teresina, Brazil), Orizon P. Ferreira (Universidade Federal de Goias, Goiania, Brazil), Arkadi Nemirovski (Georgia Tech), Takashi Tsuchiya (Institute of Statistical Mathematics in Japan), Yin Zhang (Rice University in USA), and Ming Yuan (Georgia Tech, USA).

Finally, due to his widely cited research work, the P.I. joined in 2004 the list of ISI Highly Cited Researchers (see <http://www.isihighlycited.com>).

During the duration of this project, a total of twenty two works [3, 9, 10, 14, 13, 15, 16, 19, 18, 17, 20, 21, 22, 24, 26, 27, 34] have been written which acknowledge the grant CCR-0203113. A brief description of these reports/papers mentioned above are provided below.

I. Main Research

During the course of a previous NSF grant, the P.I. in collaboration with Dr. Sam Burer and Dr. Yin Zhang have introduced a transformation [4, 7] that converts a class of semidefinite programs (SDPs) into nonlinear optimization problems free of matrix-valued constraints and variables. This transformation enables the application of nonlinear optimization techniques to the solution of certain SDPs that are too large for conventional interior-point methods to handle efficiently. Based on the transformation, a globally convergent, first-order (i.e., gradient-based) log-barrier algorithm is proposed for solving a class of linear SDPs. In paper [8], we discuss an efficient implementation of the proposed algorithm and report computational results on semidefinite relaxations of three types of combinatorial optimization problems. Our results demonstrate that the proposed algorithm is indeed capable of solving large-scale SDPs and is particularly effective for problems with a large number of constraints. This paper is related with topics 5 and 6 of the above list of goals and it partially addresses the issues raised on the Proposed Problems 9, 10, 11 and 12 of the P.I.'s research proposal.

Papers [2, 3] present first-order NLP algorithms for solving large-scale semidefinite programming (SDP) problems. The distinguishing feature of the derived algorithms is a change of variables that replaces the symmetric, positive semidefinite variable X of an SDP problem in standard form with a rectangular variable matrix R according to the factorization $X = RR^T$. The rank of the factorization, i.e., the number of columns of R , is chosen minimally so as to enhance computational speed while maintaining equivalence with the SDP problem. Fundamental results concerning the convergence of the algorithm are derived, and exceptional computational results on several large-scale test SDP problems are also presented. Our computational experiments have shown that our method substantially outperforms the other available algorithms (e.g., interior-point methods and the spectral bundle method) for solving large-scale SDP problems. For example, our method can now solve MAXCUT semidefinite relaxations of graphs containing 20,000 nodes in less than 10 minutes. On the other hand, the spectral

bundle method takes more than 60 hours to solve these problems and the interior-point methods can not even perform the first iteration due to the large size of these problems. Our algorithm has been implemented in a code called SDP-LR, which can be download from the website <http://www.isye.gatech.edu/~monteiro/software>. It is related with topics 5 and 6 of the above list of goals.

Paper [19] demonstrates that positive semidefiniteness of a large well-structured sparse symmetric matrix can be represented via positive semidefiniteness of a bunch of smaller matrices linked, in a linear fashion, to the matrix. It also derives the “dual counterpart” of the outlined representation, which expresses the possibility of positive semidefinite completion of a well-structured partially defined symmetric matrix in terms of positive semidefiniteness of a specific bunch of fully defined submatrices of the matrix. Using the representations, reformulations of well-structured large-scale semidefinite problems into smooth convex-concave saddle point problems are proposed, which are then solved by a Prox-method with efficiency $\mathcal{O}(\epsilon^{-1})$ developed in [29]. Implementations and some numerical results for large-scale Lovász capacity and MAXCUT problems are also presented.

Paper [5] proposes heuristics to approximate the stability number $\alpha(G)$ of a given graph G , i.e. the size of a maximum stable set of G . (A subset of the vertices of G forms a stable set when they are all mutually nonadjacent.) The Lovász theta number provides an upper bound on $\alpha(G)$ and can be computed in polynomial time as the optimal value of the Lovász semidefinite program. In this paper, it is shown that restricting the matrix variable in the Lovász semidefinite program to be rank-one and rank-two, respectively, yields a pair of continuous, nonlinear optimization problems each having the global optimal value $\alpha(G)$. Heuristics are then proposed for obtaining large stable sets of G based on these new formulations and computational results are presented to indicate the effectiveness of these heuristics. The code containing the implementation of the heuristics, namely Max-AO, can be downloaded from the website <http://www.isye.gatech.edu/~monteiro/software>. It is related with topic 7 of the above list of goals and it partially addresses the issues raised on the Proposed Problems 14, 15 and 16 of the P.I.’s research proposal.

Papers [25, 26, 27], written jointly with Takashi Tsuchiya, present new complexity results for interior-point algorithms for linear programming. They are related with

topic 1 of the above list of goals and partially addresses the issues raised on the Proposed Problems 1 and 7 of the P.I.'s research proposal.

Paper [25] presents a variant of Vavasis and Ye's layered-step path following primal-dual interior-point algorithm for linear programming. The algorithm is a predictor-corrector type algorithm which uses from time to time the least layered squares (LLS) direction in place of the affine scaling direction. It has the same iteration-complexity bound of Vavasis and Ye's algorithm, namely $\mathcal{O}(n^{3.5} \log(\bar{\chi}_A + n))$ where n is the number of nonnegative variables and $\bar{\chi}_A$ is a certain condition number associated with the constraint matrix A . Vavasis and Ye's algorithm requires explicit knowledge of $\bar{\chi}_A$ (which is very hard to compute or even estimate) in order to compute the layers for the LLS direction. In contrast, our algorithm uses the affine scaling direction at the current iterate to determine the layers for the LLS direction, and hence does not require the knowledge of $\bar{\chi}_A$. A variant with similar properties and with the same complexity has been developed by Megiddo, Mizuno and Tsuchiya. However, their algorithm needs to compute n LLS directions on every iteration while ours computes at most one LLS direction on any given iteration.

Paper [26] presents a new iteration-complexity bound for the Mizuno-Todd-Ye predictor-corrector (MTY P-C) primal-dual interior-point algorithm for linear programming. The analysis of the paper is based on the important notion of crossover events introduced by Vavasis and Ye. For a standard form linear program $\min\{c^T x : Ax = b, x \geq 0\}$ with decision variable $x \in \mathbb{R}^n$, we show that the MTY P-C algorithm started from a well-centered interior-feasible solution with duality gap $n\mu_0$ finds an interior-feasible solution with duality gap less than $n\eta$ in $\mathcal{O}(n^2 \log(\log(\mu_0/\eta)) + n^{3.5} \log(\bar{\chi}_A^* + n))$ iterations, where $\bar{\chi}_A^*$ is a scaling invariant condition number associated with the matrix A . More specifically, $\bar{\chi}_A^*$ is the infimum of all the conditions numbers $\bar{\chi}_{AD}$, where D varies over the set of positive diagonal matrices. Under the setting of the Turing machine model, our analysis yields an $\mathcal{O}(n^{3.5} L_A + n^2 \log L)$ iteration-complexity bound for the MTY P-C algorithm to find a primal-dual optimal solution, where L_A and L are the input sizes of the matrix A and the data (A, b, c) , respectively. In contrast, the classical iteration-complexity bound for the MTY P-C algorithm depends linearly on L instead of $\log L$.

The papers [11, 35, 36, 37] introduced a central path curvature $\nu > 0 \mapsto \kappa(\nu)$ and expressed the iteration-complexities of path-following IP algorithms for LP to reduce

the duality from $n\nu_0$ to $n\nu_1$ in terms of the curvature integral $\int_{\nu_1}^{\nu_0} \kappa(\nu)/\nu d\nu$. Intuitively, an IP path-following algorithm makes fast progress along the parts of the central path with low curvature and slow progress along the parts with large curvature. The latter observation is also exploited in the iteration-complexity analysis of the algorithms of [25, 26, 33] but from a different perspective than the one in the forementioned papers. The latter papers express the iteration-complexities of some path-following algorithms in terms of a certain condition number $\bar{\chi}_A$ associated with the LP constraint matrix A , or its scaling invariant analogue $\bar{\chi}_A^* \leq \bar{\chi}_A$. The P.I.'s paper [27] provides a first link between the above two approaches by showing that the improper integral $\int_0^\infty \kappa(\nu)/\nu d\nu$ is bounded by $\mathcal{O}(n^{3.5} \log(n + \bar{\chi}_A^*))$. It also establishes another geometric result about the central path showing that the points on the path with curvature larger than a given threshold value $\bar{\kappa} > 0$ lie in $\mathcal{O}(n^2)$ intervals, each with logarithmic length bounded by $\mathcal{O}(n \log(\bar{\chi}_A^* + n) + \log \bar{\kappa}^{-1})$. This result gives a rigorous geometric justification based on the curvature $\kappa(\cdot)$ of the central path of a claim made by Vavasis and Ye, in view of the behavior of their layered least squares path following LP method, that the path consists of $\mathcal{O}(n^2)$ long but straight continuous parts while the remaining curved portion of the path has a “logarithmic length” bounded by $\mathcal{O}(n^3 \log(\bar{\chi}_A + n))$.

Paper [20] surveys the most recent methods that have been developed for the solution of semidefinite programs. It first concentrates on the methods that have been primarily motivated by the interior point (IP) algorithms for linear programming, putting special emphasis in the class of primal-dual path-following algorithms. It then discusses methods that have been developed for solving large-scale SDP problems. These include first-order nonlinear programming (NLP) methods and more specialized path-following IP methods which use the (preconditioned) conjugate gradient or residual scheme to compute the Newton direction and the notion of matrix completion to exploit data sparsity. The P.I. gave a semi-plenary lecture at the 18th International Symposium on Mathematical Programming about the material of paper [20]. This paper is related with topic 1 of the above list of goals.

Solving systems of linear equations with “normal” matrices of the form AD^2A^T is a key ingredient in the computation of search directions for interior-point algorithms. Paper [24], establishes that a well-known basis preconditioner for such systems of linear equations produces scaled matrices with uniformly bounded condition numbers as D varies over the set of all positive diagonal matrices. In particular, it is shown that

when A is the node-arc incidence matrix of a connected directed graph with one of its rows deleted, then the condition number of the corresponding preconditioned normal matrix is bounded above by $m(n - m + 1)$, where m and n are the number of nodes and arcs of the network. This paper is related with topic 1 of the above list of goals.

Papers [17, 18, 21] deal with second-order interior-point (IP) methods based on iterative solvers. They are related with topic 1 of the above list of goals.

More specifically, papers [18, 21] consider a modified version of a well-known long-step primal-dual infeasible interior-point algorithm for solving the convex quadratic program $\min\{c^T x + x^T Q x / 2 : Ax = b, x \geq 0\}$, $A \in \mathbb{R}^{m \times n}$, where the search directions are computed by with the use of iterative linear solvers. More specifically, the preconditioner studied in [24] is used to precondition the normal coefficient matrix and the resulting preconditioned normal system of equations is solved by a standard iterative linear solver. It is shown that the number of (inner) iterations of the iterative linear solver at each (outer) iteration of the algorithm is bounded by a polynomial in m , n , and a certain condition number associated with A and Q , while the number of outer iterations is bounded by $\mathcal{O}(n^2 \log \epsilon^{-1})$, where ϵ is a given relative accuracy level. As a special case, it follows that the total number of inner iterations is polynomial in m and n for the minimum cost network flow problem with convex costs on the arcs.

Paper [17] presents a long-step primal-dual infeasible path-following algorithm for convex quadratic programming (CQP) whose search directions are computed by means of a preconditioned iterative linear solver. In contrast to the P.I.'s previous paper [18], this paper proposes a new linear system, which we refer to as the *hybrid augmented normal equation* (HANE), to determine the primal-dual search directions. Since the iterative linear solver can only generate an approximate solution to the HANE, this solution does not yield a primal-dual search direction satisfying all equations of the primal-dual Newton system. We propose a recipe to compute an inexact primal-dual search direction, based on a suitable approximate solution to the HANE. The second difference between this paper and [18] is that, instead of using the maximum weight basis (MWB) preconditioner in the above recipe for constructing the inexact search direction, this paper proposes the use of any member of a whole class of preconditioners in the above recipe, of which the MWB preconditioner is just a special case. The above proposed recipe allows us to (i) establish a polynomial bound on the number of iterations performed by our path-following algorithm and (ii) establish a uniform

bound, depending on the quality of the preconditioner, on the number of iterations performed by the iterative solver.

The conjugate gradient (CG) algorithm is well-known to have excellent theoretical properties for solving linear systems of equations $Ax = b$ where the $n \times n$ matrix A is symmetric positive definite. However, for extremely ill-conditioned matrices the CG algorithm performs poorly in practice. Paper [22] discusses an adaptive preconditioning procedure which improves the performance of the CG algorithm on extremely ill-conditioned systems. It introduces the preconditioning procedure by applying it first to the steepest descent algorithm. Then, the same techniques are extended to the CG algorithm, and convergence to an ϵ -solution in $\mathcal{O}(\log \det(A) + \sqrt{n} \log \epsilon^{-1})$ iterations is proven, where $\det(A)$ is the determinant of the matrix.

In the series of papers [10, 13, 14], the convergence of the weighted central paths associated with two central path maps and the central path for a class of degenerate semidefinite programs are studied. As a result, we obtain a new error bound that might eventually be useful in the derivation of new superlinear convergence results for interior-point primal-dual semidefinite programming methods. These papers are related with topic 2 of the above list of goals. They partially address the issues raised on the Proposed Problem 3 of the P.I.'s research proposal.

Paper [1], written jointly with Sam Burer, considers feasible long-step primal-dual path-following methods for semidefinite programming based on Newton directions associated with central path equations of the form $\Phi(PXP^T, P^{-T}SP^{-1}) - \nu I = 0$, where the map Φ and the nonsingular matrix P satisfy several key properties. An iteration-complexity bound for the long-step method is derived in terms of an upper bound on a certain scaled norm of the second derivative of Φ . As a consequence of this general framework, we derive polynomial iteration-complexity bounds for long-step algorithms based on the following four maps: $\Phi(X, S) = (XS + SX)/2$, $\Phi(X, S) = X^{1/2}SX^{1/2}$, $\Phi(X, S) = L_x^T S L_x$, and $\Phi(X, S) = W^{1/2}XSW^{-1/2}$, where L_x is the lower Cholesky factor of X and W is the unique symmetric matrix satisfying $S = W X W$. This paper is related with topic 1 of the above list of goals and it addresses the issues raised on the Proposed Problem 1 of the P.I.'s research proposal.

Paper [16] is a note which points out an error in the local quadratic convergence proof of a predictor-corrector interior-point algorithm for solving the semidefinite linear complementarity problem based on the Alizadch-Haeberly-Overton search direction

proposed in Kojima et al. [12]. A slightly modified version of the algorithm in [12] is then proposed and its local quadratic convergence is established.

Paper [15] presents a modified nearly exact (MNE) method for solving the low-rank trust region (LRTR) subproblem. The LRTR subproblem is to minimize a quadratic function, whose Hessian is a positive diagonal matrix plus an explicit low-rank update, subject to a Dikin-type ellipsoidal constraint, whose scaling matrix is positive definite and has the similar structure as the objective Hessian just described. The nearly exact (NE) method proposed by Moré and Sorensen [28] is properly modified to solve the LRTR subproblem by completely avoiding the computation of Cholesky factorizations of large-scale matrices. It is shown that the resulting MNE method is quite efficient and robust for computing NE solutions of large-scale LRTR subproblems. Moreover, a trust-region method based on this type of subproblems is developed which is able to efficiently solve large-scale nonlinear programming problems with relatively few constraints.

Paper [34] presents an application of the cone programming methodology to the area of statistics. More specifically, it deals with the statistical problem of dimension reduction and coefficient estimation in the multivariate linear model. A new method is proposed based on a novel penalized least squares formulation. The penalty employed is the coefficient matrix's Ky Fan norm. Such penalty encourages the sparsity among singular values and at the same time gives shrinkage coefficient estimates, thus conducts dimension reduction and coefficient estimation simultaneously in the multivariate linear model. The resulting least squares formulation is transformed into a semidefinite program and/or min-max convex-concave problem which is then solved by first-order methods specially tailored for this type of problems. Simulations and an application in financial econometrics demonstrate the competitive performance of the new method.

The paper [9] is less related with the topic of this project. It considers proximal point methods with Bregman distances applied to linear programming problems, and studies the dual sequence obtained from the optimal multipliers of the linear constraints of each subproblem. Convergence of this dual sequence is established, as well as convergence rate results for the primal sequence, for a suitable family of Bregman distances. These results are obtained by studying first the limiting behavior of a certain perturbed dual path and then the behavior of the dual and primal paths.

II. Other Publications

Other papers of mine, namely [4, 6, 7], have appeared in press during the course of this project but do not acknowledge the grant CCR-0203113.

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